CS 205M: Lecture 4

Comparing Infinities

and

Hopefully
(Intro to Propositional Logic)

Comparing Infinities:

Countable vs Uncountable Sets

- More precisely, we see how do we compare cardinalities ("sizes") of infinite sets.
- Infinite Sets- Sets containing infintely many distinct elements. E.g. $\mathbb{N}$, $\{x : x|2\}$ (set of even nos.).
- Comparing cardinalities of finite set: Trivial.
  To check: $|S| \leq |S'|$-
  (1) Count no. of elements in S (say k) and,
  (2) S' (say k'). Is $k \leq k'$?

Abstracting Notion of Comparing Sizes:

- Beyond the notion of counting.

"Universal" Carnival/Mela

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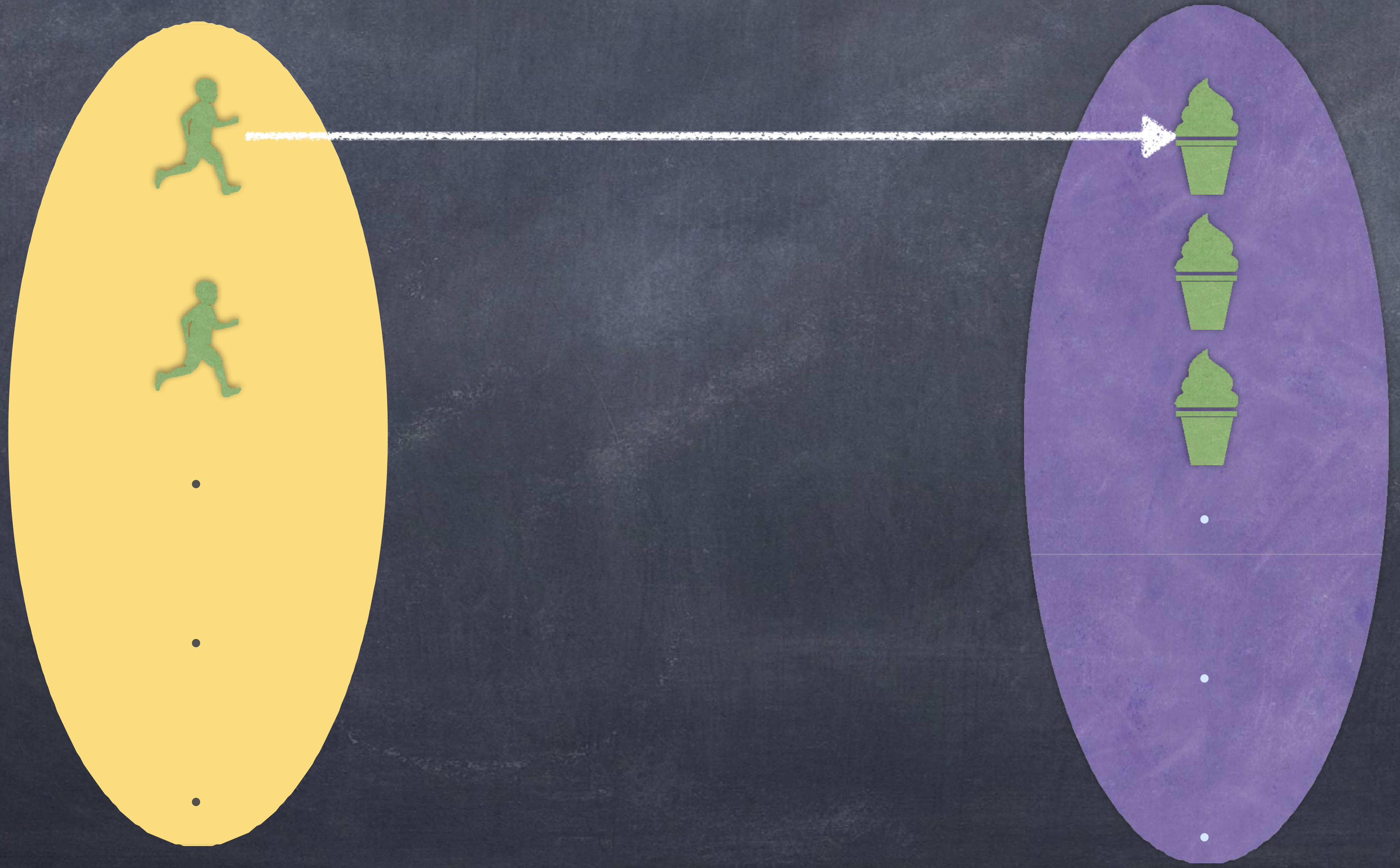

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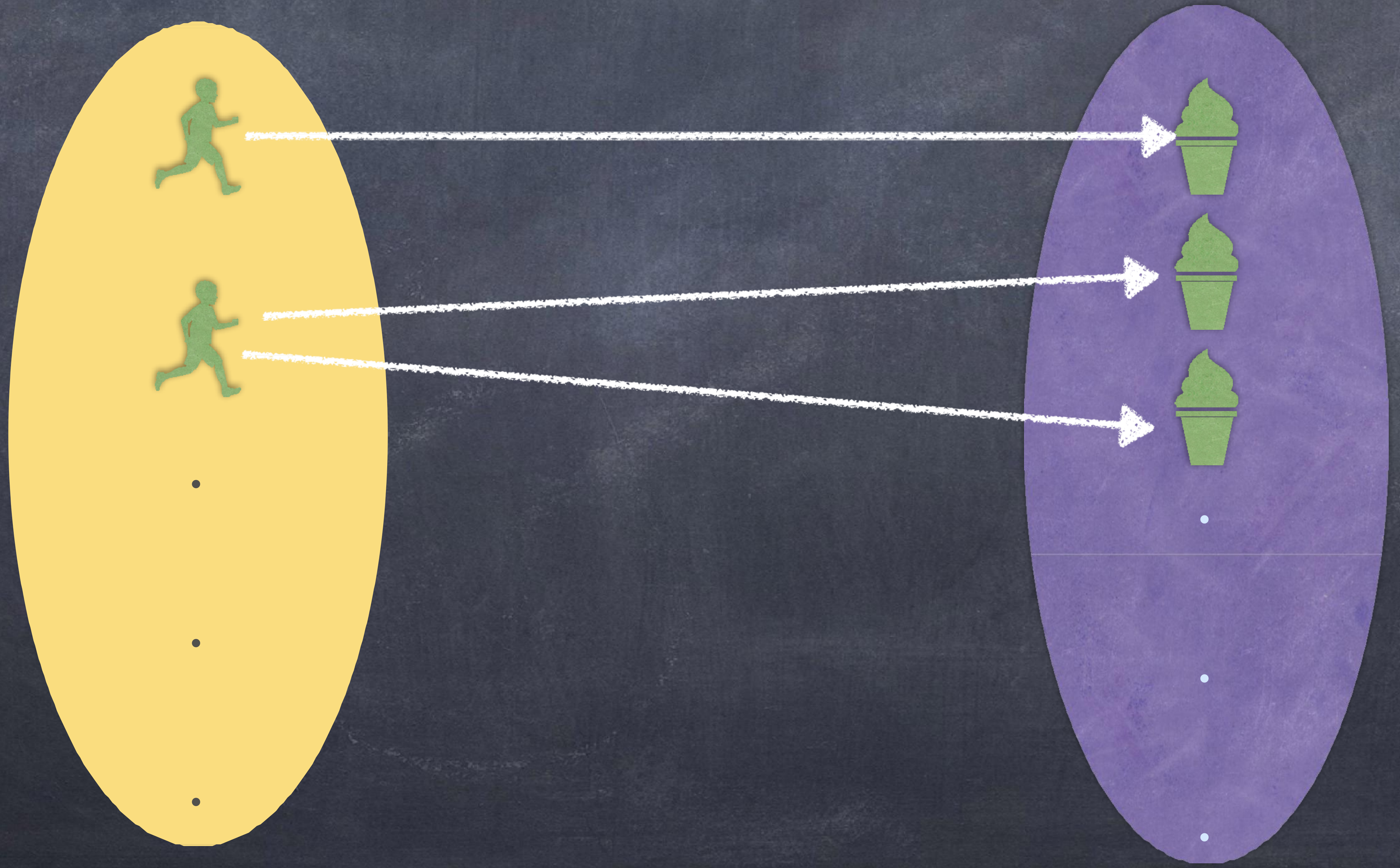

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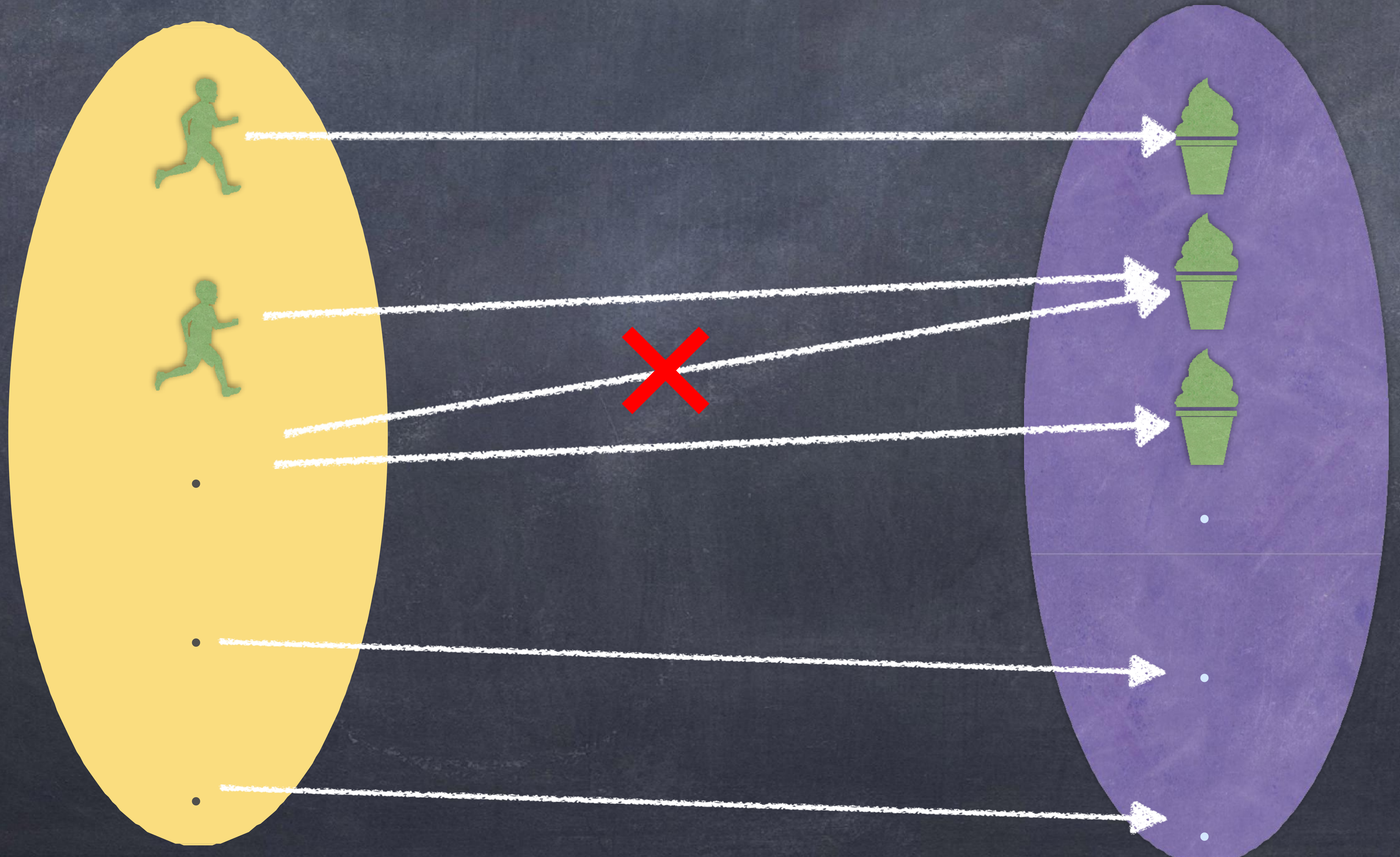
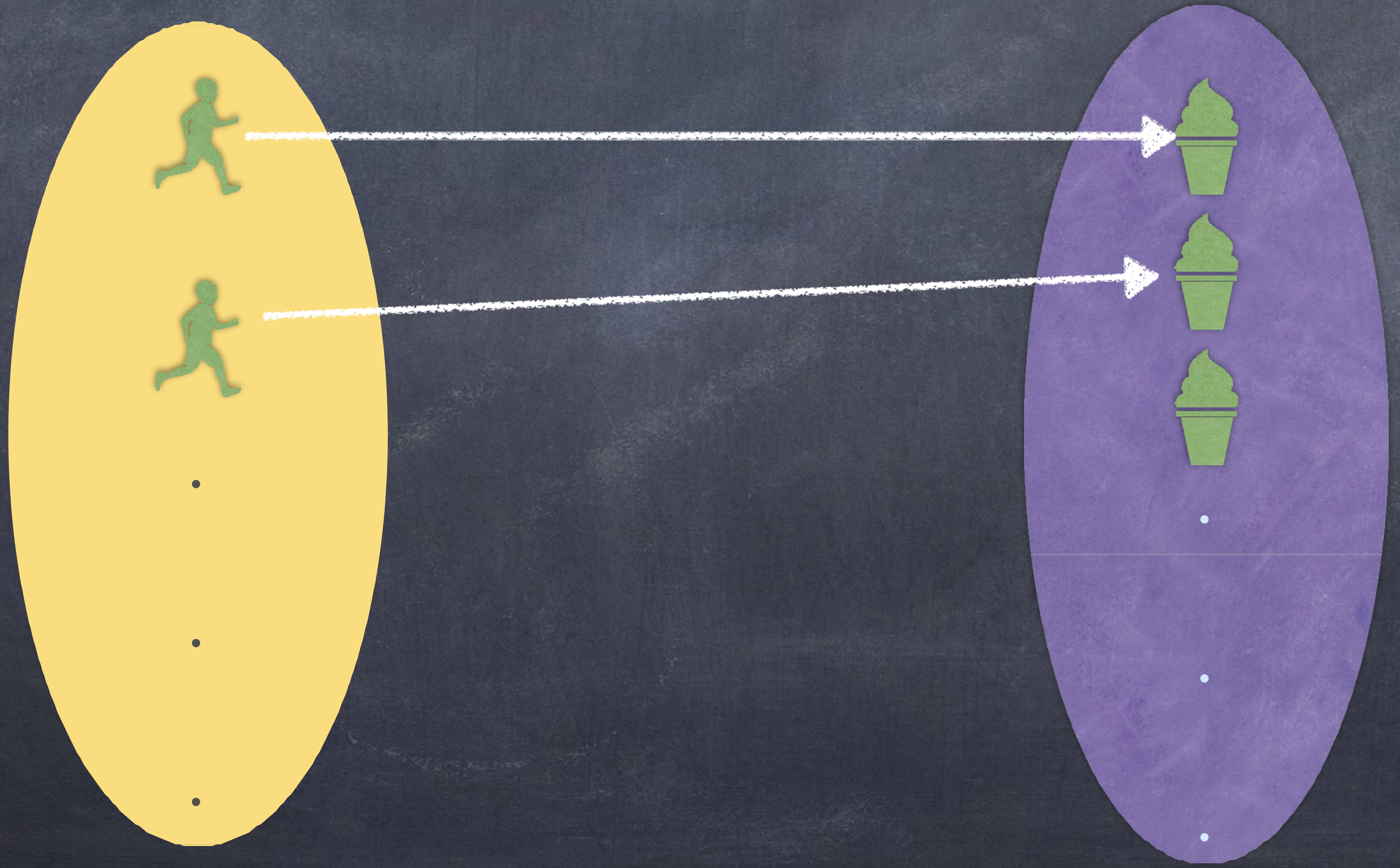


Abstracting Notion of Comparing Sizes:

- Beyond the notion of counting.

If there is an injective function from set of Kids to set of Ice-creams then

"Universal" Carnival/Mela



Abstracting Notion of Comparing Sizes:

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If there is an injective function from set of Kids to set of Ice-creams then

"Universal" Carnival/Mela

$|\text{Kids}| \leq |\text{Ice-creams}|$

Abstracting Notion of Comparing Sizes:

- Beyond the notion of counting.

If there is an injective function from set of Kids to set of Ice-creams then

"Universal" Carnival/Mela

|Kids| ≤ |Ice-creams|

|Kids| = |Ice-creams|

iff

|Kids| ≤ |Ice-creams| and |Ice-creams| ≤ |Kids|

No. of Naturals vs. No. of Rationals?

$|\mathbb{N}|<|\mathbb{Q}|$?

No. of Naturals vs. No. of Rationals?

- $|\mathbb{N}| \leq |\mathbb{Q}|$ - Trivial - Map every natural no. to itself.
- $|\mathbb{Q}| \leq |\mathbb{N}|$?
- F: $\mathbb{Q} \mapsto \mathbb{N}^2$. F$(p/q) = (p, q)$. Hence, $|\mathbb{Q}| \leq |\mathbb{N}^2|$.
- G: $\mathbb{N}^2 \mapsto \mathbb{N}$. G$((p, q)) = 2^p \times 3^q$. Notice that if $(p', q') \neq (p, q)$ then G$((p', q')) \neq$ G$((p, q))$

  (Unique Prime Factorization)
- In fact one could show that for any finite sequence of natural numbers $(a_1, \dots, a_n)$, we can associate a unique integer $2^{a_1} \times 3^{a_2} \times \dots \times k^{a_n}$, where k is the n^{th} prime.

No. of Naturals vs. No. of Reals?
$|\mathbb{N}| \leq |\mathbb{R}|$?

- $|\mathbb{N}| \leq |\mathbb{R}|$ - Trivial - Map every natural no. to itself. n_1
- $|\mathbb{R}| \leq |\mathbb{N}|$? Assume yes. This implies, F: $[0,1) \mapsto \mathbb{N}$, and F is an injective function.

n_1	$m_{1,1}$	$m_{1,2}$	$m_{1,3}$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	. . .
n_2	$m_{2,1}$	$m_{2,2}$	$m_{2,3}$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	. . .
n_3	$m_{3,1}$	$m_{3,2}$	$m_{3,3}$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	. . .
n_4	$m_{4,1}$	$m_{4,2}$	$m_{4,3}$	$m_{4,4}$	$m_{4,5}$	:	:	:	:	:	:	:	:	:	:	:	:	:	:	:	. . .
$n5$	$m_{5,1}$	$m_{5,2}$	$m_{5,3}$	$m_{5,4}$	$m_{5,5}$																. . .

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n_3	$m_{3,1}$	$m_{3,2}$	$m_{3,3}$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	. . .
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n_2	$m_{2,1}$	$m_{2,2}$	$m_{2,3}$	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	· · ·
n_3	$m_{3,1}$	$m_{3,2}$	$m_{3,3}$	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	· · ·
n_4	$m_{4,1}$	$m_{4,2}$	$m_{4,3}$	$m_{4,4}$	$m_{4,5}$	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	· · ·
n_5	$m_{5,1}$	$m_{5,2}$	$m_{5,3}$	$m_{5,4}$	$m_{5,5}$	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	· · ·

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n_5	$m_{5,1}$	$m_{5,2}$	$m_{5,3}$	$m_{5,4}$	$m_{5,5}$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	. . .

Add 1 modulo 10.

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n_2	$m_{2,1}$	$m_{2,2}$	$m_{2,3}$	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	· · ·
n_3	$m_{3,1}$	$m_{3,2}$	$m_{3,3}$	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	· · ·
n_4	$m_{4,1}$	$m_{4,2}$	$m_{4,3}$	$m_{4,4}$	$m_{4,5}$	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	· · ·
n_5	$m_{5,1}$	$m_{5,2}$	$m_{5,3}$	$m_{5,4}$	$m_{5,5}$	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	· · ·

Add 1 modulo 10.

The new no. differs from i^{th} number at i^{th} digit.

Hence, entirely new number, violating the assumtion.

In General,

- $|X| < |2^X|$, trivial for finite sets. For infinite sets,
- Let $X = \{n_1, n_2, \ldots\}$

	n_1	n_2	n_3																		
N1	$m_{1,1}$	$m_{1,2}$	$m_{1,3}$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	. . .
N2	$m_{2,1}$	$m_{2,2}$	$m_{2,3}$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	. . .
N3	$m_{3,1}$	$m_{3,2}$	$m_{3,3}$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	. . .
N4	$m_{4,1}$	$m_{4,2}$	$m_{4,3}$	$m_{4,4}$	$m_{4,5}$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	. . .
N5	$m_{5,1}$	$m_{5,2}$	$m_{5,3}$	$m_{5,4}$	$m_{5,5}$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	. . .

$m_{i,j}$s are either 0 or 1.

Flip 0s and 1s.

Why Study Logic?

- **Foundation of mathematics and computer science**
- Helps in:
  - Designing circuits
  - Writing correct software (specifications and verification)
  - Constructing valid arguments and proofs
- Used in AI, algorithms, databases, security protocols, and more

Propositions.

- **Building Blocks of Logics. That is why, it is also called as atoms.**
- Proposition: A declarative sentence that is either true or false, but not both.
- Examples:
  - "The Earth is round." ✔ (Proposition)
  - "2 + 2 = 5" ✔ (Proposition)
  - "Close the door." ✗ (Not a proposition)
  - "$x + 3 > 5$" ✗ (Depends on x; not a proposition unless x is defined)

Logical Operators: ∧, ∨, ¬

Operator	Symbol	Name
NOT	¬p	Negation
AND	p ∧ q	Conjunction
OR	p ∨ q	Disjunction

Logical Operators: $\wedge$, $\vee$, $\neg$

- Examples:
- $\neg(2 < 3) \Rightarrow$ False
- $(2 > 1) \wedge (4 = 4) \Rightarrow$ True
- $(5 > 10) \vee (3 = 3) \Rightarrow$ True

Logical Operators: ∧, ∨, ¬

- Examples:
- Conjunction (AND) – $p \wedge q$: "It is raining and it is cold.“
  True only if both are true.

- Disjunction (OR) – $p \vee q$: "I will go for a walk or I will stay home.“

  True if either (or both) happen.

- Negation (NOT) – $\neg p$: "It is not raining.“
  True if the statement "it is raining" is false.

Truth Tables.

p	q
T	T
T	F
F	T
F	F

Truth Tables.

p	q	¬p
T	T	F
T	F	F
F	T	T
F	F	T

Truth Tables.

p	q	¬p	p ∧ q
T	T	F	T
T	F	F	F
F	T	T	F
F	F	T	F

Truth Tables.

p	q	¬p	p ∧ q	p ∨ q
T	T	F	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	F

Implication: →

- If it rains, then the ground is wet.
- When is the above statement false?

Implication: →

- p → q: “If p, then q”
- False only when p is true and q is false.

p	q	p → q
T	T	T
T	F	F
F	T	T
F	F	T

Bi-implication, if and only if: →

- It rains if and only if the ground is wet.
- When is the above statement false?

Bi-implication, ↔

- p ↔ q: “p if and only if q”, p → q and q → p
- True when both p and q are same.

p	q	p ↔ q
T	T	T
T	F	F
F	T	F
F	F	T

Propositions to Bits – Bitwise Logic

- Propositions can be represented as bits: T → 1, F → 0
- Logical operations correspond to bitwise operators.

p	q	$p \wedge q$	$p \vee q$	$\neg p$
1	0	0	1	0
1	1	1	1	0

- Bitwise Ops: AND: 0101 & 1100 = 0100, OR: 0101 | 1100 = 1101,

  NOT: ~0101 = 1010 (bitwise complement)

Equivalence of Boolean Expressions:

Next Class